A card game with counter-intuitive odds

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I. STATEMENT OF THE PROBLEM

This is a variation of Mungan's two-ace problem that is meant to isolate, and hopefully demystify, some counter-intuitive concepts. See http://usna.edu/Users/physics/mungan/Scholarship/scholarship.html for Mungan's formulation. Actually, several versions of the two-ace problem have been discussed, perhaps surprisingly, with some contention in the literature [1–5]. Gardner, a veritable treasure-trove of similar puzzles, has also discussed the problem briefly in the popular press [6].

Consider four casino-style betting games, labelled $\mathbf{G}_n$ with $n \in \{1, 2, 3, 4\}$. In each game, I deal you two cards, face down, from a regular 52-card deck. I will label the suits by the index $k \in \{1, 2, 3, 4\}$, where 1 = spades, 2 = hearts, 3 = diamonds, 4 = clubs. The ace of suit k is represented by a_k. After dealing, I look at your cards make one of the following statements:

$$\begin{aligned}
s_0 &= \text{“No statement”} \\
s_1 &= \text{“You have } a_1 \text{ in your hand”} \\
s_2 &= \text{“You have } a_2 \text{ in your hand”} \\
s_3 &= \text{“You have } a_3 \text{ in your hand”} \\
s_4 &= \text{“You have } a_4 \text{ in your hand”}
\end{aligned} \tag{1}$$

In each game, each time that I make a statement other than the null statement (s_0), you are allowed to place a bet. If it turns out that you hold two aces, I pay you at 20:1, otherwise you lose your bet. My choice of statement depends on which game we are playing. The rules of $\mathbf{G}_n$ are as follows. If you have no aces in your hand, I say s_0, the null statement. If you have exactly one ace a_k satisfying $k \leq n$, then I say s_k. If you have two aces, a_j and a_k, satisfying $j, k \leq n$ then I choose between statements s_j and s_k randomly (*i.e.*, each with probability 1/2).

What is the expected return on your bet under the rules of each game? Suppose, you place a bet only if I say s_1. Is there a difference between the four games in this case? Think about your answer before you look at my solution on the next few pages!

II. PROBABILITY CALCULATIONS

The answer to the questions posed at the end of Sect I is that you will always win money in $\mathbf{G}_1$ and lose money for the other three games! Every strategy is a loser under $\mathbf{G}_{n>1}$ and a winner under $\mathbf{G}_1$. If we are playing $\mathbf{G}_n$ and I make statement s_1, the odds that you hold two aces are

$$\frac{7-n}{103-n} \tag{2}$$

Furthermore, your odds do not change when you bet only on s_1 or on any other (random or deterministic) selection of statements.

What exactly is counter-intuitive about this problem? I find it surprising that I can deal you two cards, look at both of them, then tell you truthfully that "You have the ace of spades in your hand," at which point the probability that you hold two aces depends strongly on which game we are playing (with almost a factor of 2 difference between $\mathbf{G}_1$ and $\mathbf{G}_4$). Furthermore, since the suit label is arbitrary (we could just as well interchange the names "spades" and "diamonds" if we wish), how can information about it affect the probabilities?

How do we calculate the odds that I gave above? As usual, a correct calculation is pretty much inarguable (but see [2–5] for some arguments, anyway). Since intuition is a bit suspect in this case, I will explicitly calculate some relevant probabilities and construct the solution out of these, relying heavily on Bayes' rule.

We will start with games $\mathbf{G}_1$ and $\mathbf{G}_4$ only, since these turn out to be a little easier to calculate. What is the probability that I make statement s_1 given that you have the ace of spades a_1, under the rules of these two games? We will denote these probabilities by $p_1(s_1|a_1)$ and $p_4(s_1|a_1)$, for $\mathbf{G}_1$ and $\mathbf{G}_4$, respectively. The case $n = 1$ is easy, since I make statement s_1 if and only if you have a_1:

$$p_1(s_1|a_1) = 1. \tag{3}$$

The case $n = 4$ is more interesting. Let us break all hands that include a_1 into hands that include another ace [denote these by $(a_1 a_x)$] and hands that do not include another ace [denote these by $(a_1 \overline{a_x})$]. Then we may write

$$p_4(s_1|a_1) \;=\; p_4(s_1|a_1 a_x)p(a_x|a_1) + p_4(s_1|a_1\overline{a_x})p(\overline{a_x}|a_1). \tag{4}$$

By the rules of $\mathbf{G}_4$, if you have a two-ace hand that includes a_1, that is you have a hand $(a_1 a_x)$, then the probability that I make statement s_1 is

$$p_4(s_1|a_1 a_x) = \frac{1}{2} \tag{5}$$

since it is just as likely that I make statement s_x in that case. By the $\mathbf{G}_4$ rules and simple counting, we may establish the remaining probabilities:

$$\begin{aligned} p_4(s_1|a_1\overline{a_x}) &= 1 \\ p(a_x|a_1) &= \frac{3}{51} \\ p(\overline{a_x}|a_1) &= 1 - \frac{3}{51} = \frac{48}{51}. \end{aligned} \tag{6}$$

Plugging in, we find

$$\begin{aligned} p_4(s_1|a_1) &= \frac{1}{2}\left(\frac{3}{51}\right) + \frac{48}{51} \\ &= \frac{33}{34} \approx 97\%. \end{aligned} \tag{7}$$

Let's forge ahead and calculate $p_4(s_1)$, the unconditional probability that I make statement s_1 under the rules of $\mathbf{G}_4$. Using Bayes' rule, we have

$$p_4(s_1) = \frac{p_4(s_1|a_1)p(a_1)}{p_4(a_1|s_1)}. \tag{8}$$

We just calculated $p_4(s_1|a_1)$, and we know that $p_4(a_1|s_1) = 1$ since there is no deception. Finally, $p(a_1) = 51/\binom{52}{2}$ since there are 51 two-card hands that contain a_1 and there are $\binom{52}{2}$ possible two-card hands. Combining these values, we have

$$p_4(s_1) = \frac{33}{34}\frac{51}{\binom{52}{2}} = \frac{3 \cdot 11}{2^2 \cdot 13 \cdot 17} \approx 3.7\%. \quad (9)$$

Under the rules of $\mathbf{G}_1$, we simply have

$$p_1(s_1) = p(a_1) = \frac{51}{\binom{52}{2}} = \frac{1}{2 \cdot 13} \approx 3.9\%. \quad (10)$$

I am slightly more likely to make statement s_1 under the rules of $\mathbf{G}_1$ than $\mathbf{G}_4$.

We have now computed just about everything necessary to answer the initial question (for these two games). Let's start with the easy case of $\mathbf{G}_1$. We are interested in the probability that you have two aces, given that I made statement s_1. Since s_1 is always true (under both sets of rules), we may rephrase the question to ask for the probability $p_1(a_1 a_x|s_1)$ that you have $(a_1 a_x)$ given that I said s_1 (given that I said s_1, you cannot possibly hold a two-ace hand that does not include a_1). Symbolically, we have

$$p_1(a_1 a_x|s_1) = \frac{p_1(s_1|a_1 a_x) p(a_1 a_x)}{p_1(s_1)}. \quad (11)$$

We have already computed all of these, except for $p(a_1 a_x)$, the unconditional probability that you were dealt a two-ace hand that includes a_1. Since there are only three hands of the form $(a_1 a_x)$, a simple counting argument gives

$$p(a_1 a_x) = \frac{3}{\binom{52}{2}} = \frac{1}{2 \cdot 13 \cdot 17}. \quad (12)$$

Plugging in, we find the first answer

$$p_1(a_1 a_x|s_1) = \frac{2 \cdot 13}{2 \cdot 13 \cdot 17} = \frac{1}{17}. \quad (13)$$

Moving on to the case of $\mathbf{G}_4$, we have

$$p_4(a_1 a_x|s_1) = \frac{p_4(s_1|a_1 a_x) p(a_1 a_x)}{p_4(s_1)} = \frac{\left(\frac{1}{2}\right) \frac{1}{2 \cdot 13 \cdot 17}}{\frac{3 \cdot 11}{2^2 \cdot 13 \cdot 17}} = \frac{1}{3 \cdot 11} = \frac{1}{33}. \quad (14)$$

Now let's get a bit more abstract, and calculate the odds in the general case $p_n(a_1 a_x|s_1)$. Using Bayes' rule we may write

$$p_n(a_1 a_x|s_1) = \frac{p_n(s_1|a_1 a_x) p(a_1 a_x)}{p_n(s_1)}. \quad (15)$$

Using some symbolic manipulation (and relying on the exclusivity of each hand), we have

$$p_n(a_1 a_x|s_1) = \frac{\left[\sum_{k=2}^n p_n(s_1|a_1 a_k) p(a_1 a_k)\right] / p(a_1 a_x)}{\left[\sum_{k=2}^n p_n(s_1|a_1 a_k) p(a_1 a_k)\right] + p_n(s_1|a_1 \overline{a_x}) p(a_1 \overline{a_x})} p(a_1 a_x) = \frac{1}{1 + \xi_n} \quad (16)$$

where

$$\xi_n = \frac{p(s_1|a_1\overline{a_x})p(a_1\overline{a_x})}{\sum_{k=2}^{n} p_n(s_1|a_1a_k)p(a_1a_k)}. \tag{17}$$

Note that, according to the rules, $p_n(s_1|a_1a_k) = 1/2$ for $k \leq n$ and 1 otherwise. Similarly, $p_n(s_1|a_1\overline{a_x}) = 1$. Plugging in, we then have

$$\begin{aligned} \xi_n &= \frac{48/\binom{52}{2}}{\left[(n-1)\left(\frac{1}{2}\right) + 3 - (n-1)\right]/\binom{52}{2}} \\ &= \frac{96}{7-n} \end{aligned} \tag{18}$$

giving the general result

$$p_n(a_1a_x|s_1) = \frac{7-k}{103-k} \tag{19}$$

as mentioned at the start of the section. Your odds just get worse, as my statements become more ambiguous:

$$\begin{aligned} p_1(a_1a_x|s_1) &= \tfrac{1}{17} \approx 5.9\% \\ p_2(a_1a_x|s_1) &= \tfrac{5}{101} \approx 5.0\% \\ p_3(a_1a_x|s_1) &= \tfrac{1}{25} \approx 4.0\% \\ p_4(a_1a_x|s_1) &= \tfrac{1}{33} \approx 3.0\%. \end{aligned}$$

So it is true that, even conditioned on exactly the same statement, s_1, the odds that you hold two aces depend (strongly) on the rules of the game we are playing!

Finally, at 20:1 odds, when you bet on s_1 your expected winnings E_n under $\mathbf{G}_n$ are given by

$$\begin{aligned} E_n &= -1\left[1 - p_n(a_1a_x|s_1)\right] + 20p_n(a_1a_x|s_1) \\ &= 6\frac{5-3n}{103-n} \end{aligned} \tag{20}$$

However, every game is symmetric with respect to suits *for which I am allowed to make a statement*, so your odds of winning are the same every time I make any statement (other than s_0 of course). The expected winning change from positive to negative at $n = 5/3$, so that you win for $n = 1$ and lose for $n \geq 2$. Therefore, as advertised at the start of the section, every strategy is a loser under $\mathbf{G}_{n>1}$ and a winner under $\mathbf{G}_1$.

III. DISCUSSION

Now that we have calculated the odds in excruciating detail, lets try to gain some intuition. Those prime numbers (7 and 103) in Eq. (19) are a bit strange. Perhaps it helps to rewrite (19) as

$$p_n(a_1a_x|s_1) = \frac{3 - \left(\frac{n-1}{2}\right)}{51 - \left(\frac{n-1}{2}\right)} \tag{21}$$

since 3 and 51 are numbers we expect in the solution (3 is just the number of remaining aces and 51 is the number of remaining cards, given s_1). Dare I say that equation (21) gives us some intuition into the problem? Consider $\mathbf{G}_2$. Given statement s_1, you know that you have at least a_1. However, the probability that you have (a_1a_2) is reduced by 1/2 relative to all other hands of the form $(a_1\overline{a_2})$ (where the bar means "not" as before). So in a loose sense, (a_1a_2) only counts as "half" a hand when you add up your probabilities of winning. Of course your total possible number of hands is also reduced by 1/2 by the same argument, so we expect $p_2(a_1a_x|s_1) = (3-1/2)/(51-1/2)$. This argument is presented graphically in Fig. 1.

In Mungan's statement of the problem, he does not consider betting games. He simply asks (I'm paraphrasing here), "What is the probability that you hold a two-ace hand, given that one of your cards is an ace?" and "What is the probability that you hold a two-ace hand, given that one of your cards is the ace of spades?". He solves these problems (1/33 and 1/17 respectively) using counting arguments. Which game are we playing in Mungan's problem? From the odds, we suspect a correspondence between the former question and $\mathbf{G}_4$ and a similar correspondence between the

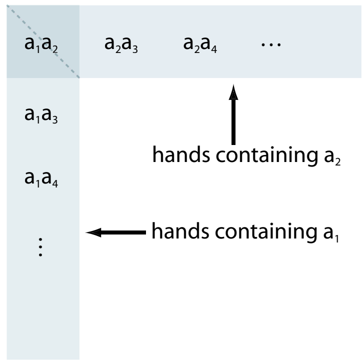

FIG. 1: Diagram showing the sets of hands that containing a_1 and a_2. Their overlap is the hand (a_1a_2). Under $\mathbf{G}_2$, the statement s_1 conditionally reduces the probability of this hand (its area in the diagram) to 1/2 relative to all other hands containing a_1. The probability that you win is the ratio of the area of winning hands to the total area of the set. Conditioning on s_1 *or* s_2 does not change the ratio of winning to losing hands, even though it "restores" the area of (a_1a_2), since it also adds a symmetric area of losing hands . The same argument holds for all games $\mathbf{G}_n$ but the corresponding diagram becomes harder to draw.

latter question and $\mathbf{G}_1$. Can we place this correspondence on a concrete footing? Under $\mathbf{G}_4$, you learn suit information about your hand *if and only if* you hold an ace (any ace). Therefore, by the symmetry of $\mathbf{G}_4$, every statement (s_1 through s_4) carries the same amount of information as the statement "you have at least one ace." Therefore, your odds of winning are 1/33 in this case. When you are told you have the ace of spades, but with no other context (such as the rules of $\mathbf{G}_n$), then you must assume the *maximum entropy* distribution over the remaining allowed hands. You must therefore assume that the conditional probability that you hold any of the 51 hands containing the ace of spades (the conditionally allowed hands) is a flat distribution. Your odds of winning are therefore just the fraction of allowed hands which are winners. Therefore, you calculate your odds to be 3/51=1/17, just as in $\mathbf{G}_1$ where the probability distribution over allowed hands is flat.

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